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Reflection from Inhomogeneous Plane Stratified Media

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REFLECTION FROM INHOMOGENEOUS PLANE STRATIFIED MEDIA

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Abstract

The problem of computing the reflected field due to the incidence of a plane electromagnetic wave on a plane stratified inhomogeneous medium is considered. An approximate solution to this problem is examined, and the error is estimated. It is found that when the medium is slowly varying, in a sense that can be defined precisely, the error is small. It can also be seen that the error will be small in this case no matter what the wavelength may be.

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1. Introduction

This paper is concerned with the problem of calculating the electromagnetic reflection properties of a plane stratified inhomogeneous dielectric medium such as an atmospheric layer of the type indicated by certain microwave refractometer measurements. We shall assume that the electrical properties of the medium do not vary greatly, which is apparently the case in an atmospheric layer, but we shall allow the medium to have a variable finite conductivity since no essential complication occurs when we do so. The relations between the electrical parameters are assumed to be such that the phenomenon of total reflection cannot take place.

An approximate method for calculating the reflected wave due to an incident plane wave will be described, and the error committed in the use of the approximation will be estimated. We shall show that this approximation is correct for all wavelengths to within the error estimate which is independent of the wavelength. Moreover, for the validity of the method, the variation of the medium is not required to be smooth but only continuous with piecewise continuous derivatives.

The approximation we shall use has been given in one form or another by several authors. It is a special case of the method devised by J. B. Keller and H. B. Keller[1] for a system of differential equations. The same approximation was given by Bremmer[2] and by Schelkunoff[3], and it is derived in the thesis of Landau[4]. The approximation is often regarded as a correction to classical optics in one dimension or the W.K.B. solution[2], hence the implication that its accuracy is frequency dependent. We shall see, however, that while the accuracy may increase with increasing frequency the error can be bounded by a small number which is independent of the frequency.

This paper is divided into sections so that a part or all of the analysis can be easily skipped, if desired. In fact, Section 7 contains a summary of the results necessary for applications.

2. Statement of the mathematical problem

The description of our problem will be given in terms of a Cartesian coördinate system which has its z-axis oriented vertically so that z increases in the upward direction. The reflecting medium is assumed to be stratified in horizontal layers, all electrical properties of the medium being constant in any plane parallel to the x,y-plane of the coördinate system.

Since the electrical properties of the medium can vary in the z direction only, the dielectric parameter $\varepsilon(z)$, the permeability $\mu(z)$ and the conductivity $\sigma(z)$ can be functions of z but not of x and y. It is assumed that the variable part of the medium has a lower boundary, the x,y-coördinate plane. Thus, ε, μ and σ are all constant for $z < 0$. We designate the electrical parameters for the homogeneous medium in the region $z < 0$ by ε_0, μ_0, σ_0. The inhomogeneous part of the medium either extends upward to infinity or else it is bounded by a plane $z = \triangle$ some positive distance $\triangle$ above its lower boundary at $z = 0$. For $z > \triangle$ the medium is assumed to be homogeneous again.

A plane electromagnetic wave of arbitrary polarization is incident at an angle α from the homogeneous region $z < 0$. The coordinate system is further oriented so that the y,z-coördinate plane is the plane of incidence.

Maxwell's equations for a wave of circular frequency ω will be used in M.K.S. units:

$$(2.1) \qquad \begin{aligned} \nabla \times E &= -j\omega\mu H \\ \nabla \times H &= (\sigma + j\omega\varepsilon)E = j\omega\hat{\varepsilon}\, E, \end{aligned}$$

where $\hat{\varepsilon} = (\sigma + j\omega\varepsilon)/j\omega$, and the time factor has been taken to be $\exp(j\omega t)$.

It is convenient to think of the electromagnetic field as the sum of two separate fields: one with the E vector polarized parallel to the plane of incidence, and one polarized normal to the plane of incidence. Thus, the field represented by the pair E, H is a sum of the fields represented by

PLANE WAVE INCIDENT ON A STRATIFIED LAYER

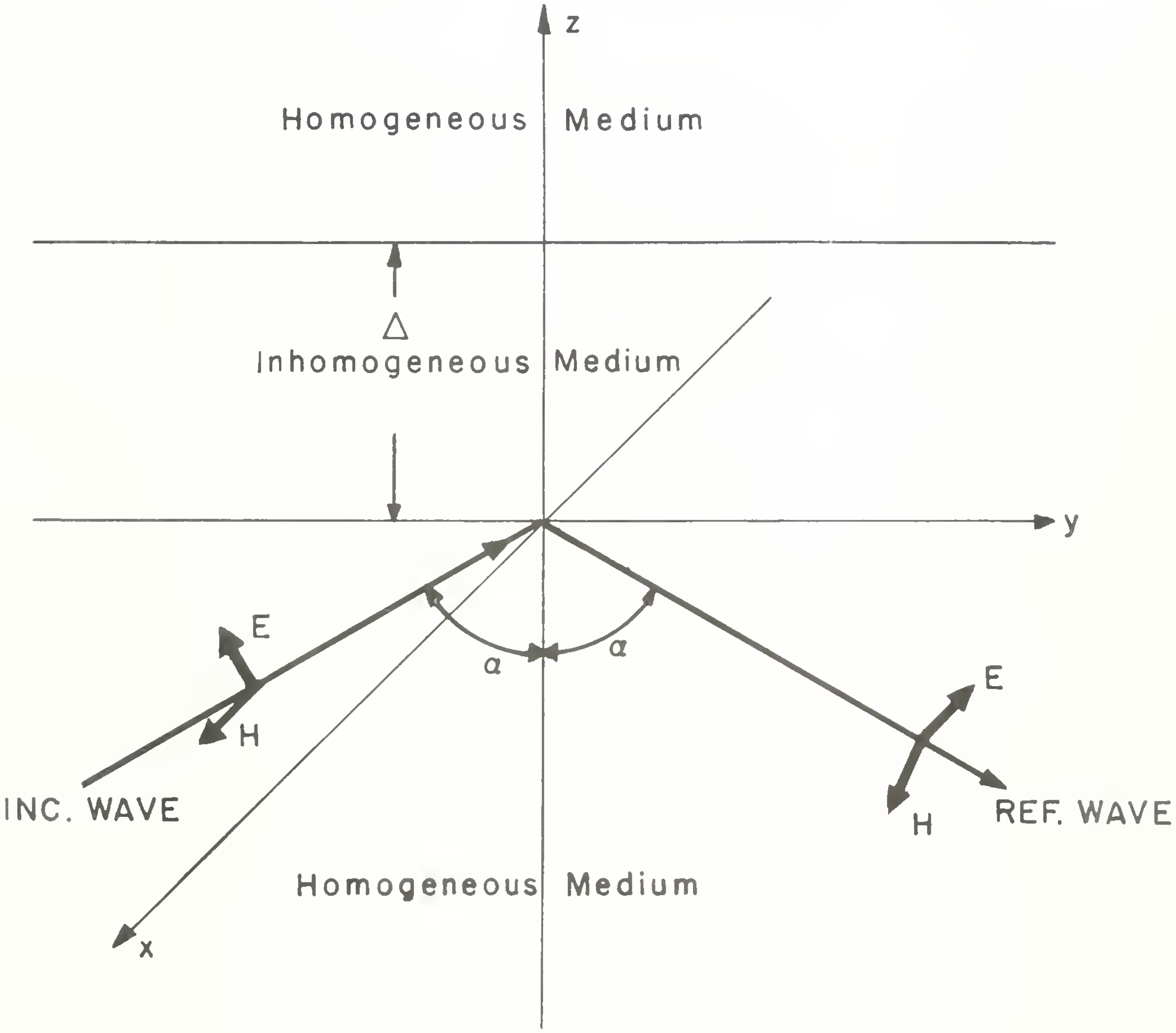

Figure 1

the pairs E_p, H_p and E_n, H_n, where the subscript p indicates polarization parallel to the plane of incidence and n indicates polarization normal to the plane of incidence. Since the y,z-coördinate plane is the plane of incidence and α is the angle of incidence we have:

$$(2.2) \qquad E_p = (0, V_p, \mathcal{E})\exp(-jk_o y \sin \alpha), \qquad H_p = (I_p, 0, 0)\exp(-jk_o y \sin \alpha)$$

$$E_n = (V_n, 0, 0)\exp(-jk_o y \sin \alpha), \qquad H_n = (0, I_n, \ell)\exp(-jk_o y \sin \alpha),$$

where $k_o = \omega\sqrt{\varepsilon_c \mu_o}$ is a propagation constant referred to the homogeneous medium of the region $z < 0$.

From these assumed forms for E_p, H_p, E_n, H_n, by substituting into (2.1), we obtain the two pairs of transmission line equations:

$$(2.3) \qquad dV_i/dz = -j\omega L_i I_i$$

$$dI_i/dZ = -j\omega C_i V_i, \qquad i = p,n$$

with $L_p = (\mu_o \varepsilon_o \sin^2 \alpha)/\hat{\varepsilon} - \mu$, $C_p = -\hat{\varepsilon}$, $L_n = \mu$, $C_n = \hat{\varepsilon} - (\mu_o \varepsilon_o \sin^2 \alpha)/\mu$. We also have the auxiliary equations for the other components:

$$(2.4) \qquad \mathcal{E}(z) = \left\{(k_o \sin \alpha)/\omega\hat{\varepsilon}\right\} I_p(z), \qquad \ell(z) = \left\{(-k_o \sin \alpha)/\omega\mu\right\} V_n(z).$$

The quantity $L_i(z)$ may be thought of as a distributed inductance and $C_i(z)$ as a distributed capacitance, whence it is natural to define a characteristic or wave impedance,

$$(2.5) \qquad \zeta_i(z) = \left\{L_i(z)/C_i(z)\right\}^{1/2}, \qquad i = p,n$$

and a wave velocity

$$(2.6) \qquad v_i(z) = \left\{L_i(z)C_i(z)\right\}^{-1/2}, \qquad i = p,n.$$

Corresponding to the wave velocity of (2.6) we have a wave number

$$(2.7)\qquad k_i(z) = \omega/v_i(z), \qquad i = p,n.$$

Using these definitions we have in place of (2.3)

$$(2.8)\qquad \begin{aligned} dV_i/dz &= -j\left\{k_i(z)\zeta_i(z)\right\} I_i \\ dI_i/dz &= -j\left\{k_i(z)/\zeta_i(z)\right\} V_i, \qquad i = p,n. \end{aligned}$$

Since the quantities V_i and I_i are solutions of a pair of linear differential equations, they may be regarded as the components of a two-dimensional vector (V_i,I_i). In order to describe the behavior of traveling waves it is convenient to use a representation of the vector (V_i,I_i) different from that implied by the transmission line equations (2.3). The new vector representation will be connected to the old one by a linear transformation:

$$(2.9)\qquad \begin{aligned} V_i(z) &= U_i(z) + D_i(z) \\ I_i(z) &= \left\{U_i(z) - D_i(z)\right\} / \zeta_i(z), \qquad i = p,n, \end{aligned}$$

where the quantities $U_i(z)$ and $D_i(z)$ are the components of the transformed vector (U_i,D_i). The transmission line equations (2.8) become:

$$(2.10)\qquad \begin{aligned} dU_i/dz + \left\{jk_i - (d\zeta_i/dz)/2\zeta_i\right\} U_i &= \left\{-(d\zeta_i/dz)/2\zeta_i\right\} D_i \\ dD_i/dz + \left\{-jk_i - (d\zeta_i/dz)/2\zeta_i\right\} D_i &= \left\{-(d\zeta_i/dz)/2\zeta_i\right\} U_i, \qquad i = p,n. \end{aligned}$$

If we assume that the quantity $d\zeta_i/dz$ is small (in some sense to be specified more precisely later on), we should expect from the form of (2.10) that U_i represents a wave traveling upward only while D_i represents a wave traveling downward only. It is clear then that the quantities U_i and D_i are more appropriate for describing traveling waves than the original quantities V_i and I_i.

3. Approximate solution

In view of the assumption that $d\zeta_i/dz$ is to be small it is natural to attempt to solve equations (2.10) by means of a perturbation procedure. If the right side of equations (2.10) were to vanish, a solution for $z < \Delta$, except for two arbitrary constant factors, would be:

$$(3.1) \qquad \begin{aligned} U_i(z) &= \zeta_i^{1/2}(z) \exp\left\{ j \int_z^{\Delta} k_i(s)ds \right\} \\ D_i(z) &= \zeta_i^{1/2}(z) \exp\left\{ -j \int_z^{\Delta} k_i(s)ds \right\}, \qquad i = p,n. \end{aligned}$$

This fact suggests a change of variables from the vector (U_i, D_i) to the perturbation vector (u_i, d_i):

$$(3.2) \qquad \begin{aligned} U_i(z) &= u_i(z)\zeta_i^{1/2}(z) \exp\left\{ j \int_z^{\Delta} k_i(s)ds \right\} \\ D_i(z) &= d_i(z)\zeta_i^{1/2}(z) \exp\left\{ -j \int_z^{\Delta} k_i(s)ds \right\}, \qquad i = p,n. \end{aligned}$$

Equations (2.10) then become:

$$(3.3) \qquad \begin{aligned} u_i'(z) &= \left\{ -\zeta_i'(z)/2\zeta_i(z) \right\} \exp\left\{ -2j \int_z^{\Delta} k_i(s)ds \right\} d_i(z) \\ d_i'(z) &= \left\{ -\zeta_i'(z)/2\zeta_i(z) \right\} \exp\left\{ 2j \int_z^{\Delta} k_i(s)ds \right\} u_i(z), \qquad i = p,n, \end{aligned}$$

where primes denote differentiation as usual. Equations (3.3) have solutions which are constant if the medium is homogeneous.

In order to approximate the solution to (3.3) by means of a perturbation procedure, it is helpful to change (3.3) to an integral equation by integrating both sides. We have:

$$(3.4)\qquad u_i(z) = u_i(\Delta) + \int_z^{\Delta} \left\{ \zeta_i'(y)/2\zeta_i(y) \right\} \exp\left\{ -2j \int_y^{\Delta} k_i(s)ds \right\} d_i(y)dy$$

$$d_i(z) = \int_z^{\Delta} \left\{ \zeta_i'(y)/2\zeta_i(y) \right\} \exp\left\{ 2j \int_y^{\Delta} k_i(s)ds \right\} u_i(y)dy, \qquad i = p,n,$$

where we have used the fact that only the upgoing wave emerges from the top of the inhomogeneous layer, and therefore $d_i(\Delta) = 0$.

We may obtain an approximate solution to the integral equations (3.4) by a standard iteration procedure. A single iteration can be obtained by substituting the inhomogeneous term $u_i(\Delta)$ and the term $d_i(\Delta)$ which is zero in (3.4) into the integrals in place of the solution functions $u_i(y)$ and $d_i(y)$. We obtain, after replacing $u_i(y)$ by $u_i(\Delta)$ and $d_i(y)$ by zero:

$$(3.5)\qquad u_i(z) \sim u_i(\Delta)$$

$$d_i(z) \sim u_i(\Delta) \int_z^{\Delta} \left\{ \zeta_i'(y)/2\zeta_i(y) \right\} \exp\left\{ 2j \int_y^{\Delta} k_i(s)ds \right\} dy, \qquad i = p,n.$$

4. Iteration of a general system of integral equations

Consider a general system of integral equations of which (3.4) is a special case:

$$(4.1)\qquad w_i(z) = \overset{o}{w}_i(z) + \int_z^{\Delta} \sum_{j=1}^{n} A_{ij}(y) w_j(y)dy, \qquad i = 1,\dots,n.$$

Let $\overset{m}{w}_i(z)$ be the m-th approximation of an iteration procedure for solving equations (4.1). The iteration procedure is described by the recursion relation:

$$(4.2) \qquad \overset{m}{w}_i(z) = \overset{o}{w}_i(z) + \int_z^{\Delta} \sum_{j=1}^{n} A_{ij}(y)\, \overset{m-1}{w}_j(y)dy, \qquad i = 1,\dots,n.$$

The solution of (4.1) is given by the limit

$$w_i(z) = \lim_{m \to \infty} \overset{m}{w}_i(z),$$

as one can show, using a standard method. The error committed by stopping at a finite value of m can be estimated at the same time, as the following demonstration shows.

From the recursion formula (4.2) we obtain a recursion formula for the difference between two successive terms of the sequence:

$$(4.3) \qquad \overset{m}{w}_i(z) - \overset{m-1}{w}_i(z) = \int_z^{\Delta} \sum_{j=1}^{n} A_{ij}(y) \left\{ \overset{m-1}{w}_j(y) - \overset{m-2}{w}_j(y) \right\} dy, \qquad i = 1,\dots,n.$$

In order to estimate the error we shall need a bound on the effect of the functions $A_{ij}(y)$ in (4.3). We therefore define:

$$(4.4) \qquad \mu(y) = \left\{ \sum_{j=1}^{n} \left[\sum_{i=1}^{n} |A_{ij}(y)| \right]^2 \right\}^{1/2} .$$

We must obtain a single estimate for the largest error among the w_i.

From (4.3) we have

$$(4.5) \quad \left\{ \sum_{i=1}^{n} \left| \overset{m}{w}_i(z) - \overset{m-1}{w}_i(z) \right|^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{i=1}^{n} \left| \int^{\Delta} \sum_{j-1}^{n} A_{ij}(y) \left[\overset{m-1}{w}_j(y) - \overset{m-2}{w}_j(y) \right] dy \right|^2 \right\}^{\frac{1}{2}}$$

$$< \sum_{i=1}^{n} \left| \int^{\Delta} \sum_{j=1}^{n} A_{ij}(y) \left\{ \overset{m-1}{w}_j(y) - \overset{m-2}{w}_j(y) \right\} dy \right| \leq \int_z^{\Delta} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} |A_{ij}(y)| \right\} \left| \overset{m-1}{w}_j(y) - \overset{m-2}{w}_j(y) \right| dy$$

$$\leq \int^{\Delta} \left\{ \sum_{j=1}^{n} \left[\sum_{i=1}^{n} |A_{ij}(y)| \right]^2 \right\}^{1/2} \left\{ \sum_{j=1}^{n} \left| \overset{m-1}{w}_j(y) - \overset{m-2}{w}_j(y) \right|^2 \right\}^{1/2} dy$$

$$= \int_z^{\Delta} \mu(y) \left\{ \sum_{j=1}^{n} \left| \overset{m-1}{w}_j(y) - \overset{m-2}{w}_j(y) \right|^2 \right\}^{1/2} dy .$$

With inequality (4.5) we can obtain an estimate of the difference between two successive terms of the iteration sequence. We have by (4.5) and by induction:

$$(4.6) \qquad \left\{\sum_{i=1}^{n}\left|\overset{m}{w}_i(z)-\overset{m-1}{w}_i(z)\right|^2\right\}^{1/2} < W\int_z^{\triangle}\mu(y_m)dy_m\int_{y_m}^{\triangle}\mu(y_{m-1})dy_{m-1}\cdots\int_{y_2}^{\triangle}\mu(y_1)dy_1,$$

where $W = \max\limits_{z \le y \le \triangle}\left\{\sum\limits_{i=1}^{n}\left|\overset{o}{w}_i(y)\right|^2\right\}^{1/2}$. Through successive integrations by parts the right side of (4.6) can be put into a simpler form. Setting

$$S(y) = \int_z^{y}\mu(s)ds,$$

we have

$$(4.7) \qquad W\int_z^{\triangle}\mu(y_m)dy_m\int_{y_m}^{\triangle}\mu(y_{m-1})dy_{m-1}\cdots\int_{y_2}^{\triangle}\mu(y_1)dy_1$$

$$= W\int_z^{\triangle}\mu(y_{m-1})S(y_{m-1})dy_{m-1}\int_{y_{m-1}}^{\triangle}\mu(y_{m-2})dy_{m-2}\cdots\int_{y_2}^{\triangle}\mu(y_1)dy_1$$

$$= (W/2)\int_z^{\triangle}\left\{d\left[S^2(y_{m-1})\right]/dy_{m-1}\right\}dy_{m-1}\int_{y_{m-1}}^{\triangle}\mu(y_{m-2})dy_{m-2}\cdots\int_{y_2}^{\triangle}\mu(y_1)dy_1$$

$$= (W/3!)\int_z^{\triangle}\left\{d\left[S^3(y_{m-2})\right]/dy_{m-2}\right\}dy_{m-2}\int_{y_{m-2}}^{\triangle}\mu(y_{m-3})dy_{m-3}\cdots\int_{y_2}^{\triangle}\mu(y_1)dy_1$$

$$= (W/m!)\,S^m(\triangle) = (W/m!)\left\{\int_z^{\triangle}\mu(s)ds\right\}^m .$$

We can now demonstrate that the sequence $\overset{m}{w}_i(z)$ actually converges. We have

$$\overset{N}{w}_i(z) - \overset{o}{w}_i(z) = \sum_{m=0}^{N-1} \left\{ \overset{m+1}{w}_i(z) - \overset{m}{w}_i(z) \right\} .$$

Thus,

$$(4.8) \qquad \overset{\infty}{w}_i(z) - \overset{o}{w}_i(z) = \lim_{N \to \infty} \left\{ \overset{N}{w}_i(z) - \overset{o}{w}_i(z) \right\} = \sum_{m=o}^{\infty} \left\{ \overset{m+1}{w}_i(z) - \overset{m}{w}_i(z) \right\} ,$$

providing the limit exists, of course. We have

$$\left| \overset{m}{w}_i(z) - \overset{m-1}{w}_i(z) \right| \leq \left\{ \sum_{i=1}^{m} \left| \overset{m}{w}_i(z) - \overset{m-1}{w}_i(z) \right|^2 \right\}^{1/2} .$$

Thus, by (4.6) and (4.7), since

$$(4.9) \qquad \sum_{m=o}^{\infty} (W/m!) \left\{ \int_z^{\Delta} \mu(s)ds \right\}^m = W \left\{ \exp\left[\int_z^{\Delta} \mu(s)ds \right] \right\} ,$$

exists, the limit in (4.8) exists, and the sequence $\overset{m}{w}_i(y)$ actually converges uniformly in the interval $z \leq y \leq \Delta$. From this fact and (4.2) it follows that

$$\lim_{m \to \infty} \overset{m}{w}_i(z) = \overset{\infty}{w}_i(z) = w_i(z),$$

where $w_i(z)$ is a solution of (4.1). There is no point in considering the question of uniqueness here.

We are interested most in the error committed by stopping the sequence at a finite value of m. It is obvious from the preceding paragraph that this error is less than the error which would occur in (4.9) if the infinite series on the left were terminated at the given finite value of m. We have for the error $\overset{m}{\varepsilon}$:

$$(4.10) \qquad \overset{m}{\varepsilon}_i = w_i(z) - \overset{m}{w}_i(z) = \sum_{j=o}^{\infty} \left\{ \overset{m+j+1}{w}_i(z) - \overset{m+j}{w}_i(z) \right\} , \qquad i = 1,\ldots,n.$$

By (4.10) and a repeated application of the triangle inequality we have

$$(4.11) \qquad \varepsilon_m = \max_{1 \leq i \leq n} \left| \overset{m}{\varepsilon}_i \right| \leq \left\{ \sum_{i=1}^{n} \left| w_i(z) - \overset{m}{w}_i(z) \right|^2 \right\}^{1/2}$$

$$\leq \sum_{j=o}^{\infty} \left\{ \sum_{i=1}^{n} \left| \overset{m+j+1}{w}_i(z) - \overset{m+j}{w}_i(z) \right|^2 \right\}^{1/2} .$$

We combine (4.6), (4.7), (4.9) and (4.11) with the result that

$$(4.12) \qquad \varepsilon_m \leq W \left\{ \exp\left[\int_z^{\triangle} \mu(s)ds \right] - 1 - \sum_{\nu=1}^{m} \left[\int_z^{\triangle} \mu(s)ds \right]^{\nu} / \nu ! \right\} ,$$

which is our estimate of the error committed when the sequence is stopped at the m-th term.

5. Application to the error estimate for the electromagnetic problem

We are interested in a special case of the general system treated in Section 4, namely the integral equations (3.4) associated with the electromagnetic problem for plane stratified media discussed in Section 2. We can thus identify the important quantities of Sections 2 and 3 with the various quantities of Section 4 in this case:

$$w_1(z) = u_i(z), \qquad w_2(z) = d_i(z), \qquad \overset{o}{w}_i(z) = u_i(\triangle), \quad \overset{o}{w}_2(z),$$

$$A_{11}(z) = A_{22}(z) = 0, \quad A_{12} = \left\{ \zeta_i'(z)/2\zeta_i(z) \right\} \exp \left\{ -2j \int_z^{\triangle} k_i(s)ds \right\} ,$$

$$A_{21}(z) = \left\{ \zeta_i'(z)/2\zeta_i(z) \right\} \exp \left\{ 2j \int_z^{\triangle} k_i(s)ds \right\} .$$

Thus, by (4.4) and (4.6)

$$(5.1) \qquad \mu(z) = (1/\sqrt{2}) \left| \zeta_i'(z)/\zeta_i(z) \right| ,$$

$$W = |u_i(\triangle)| \quad \text{and} \quad m = 1.$$

Thus, by (4.12), when (3.5) is our approximate solution the error is less than ε, where

$$(5.2) \qquad \varepsilon \leq \left|u_i(\triangle)\right| \left\{ \exp\left[(1/\sqrt{2})\int_z^{\triangle} |\zeta_i'(y)/\zeta_i(y)|\,dy\right] - 1 - (1/\sqrt{2})\int_z^{\triangle} |\zeta_i'(y)/\zeta_i(y)|\,dy \right\} .$$

The integral in (5.2) can be evaluated more explicitly. We have

$$\zeta_i = |\zeta_i| \exp(j\, \phi_i),$$

where $\phi_i(y)$ is the phase of $\zeta_i(y)$. Thus,

$$\zeta_i'/\zeta_i = |\zeta_i|' / |\zeta_i| + j\, \phi_i' ,$$

and therefore

$$|\zeta_i' / \zeta_i| \leq \left||\zeta_i|' / |\zeta_i|\right| + |\phi_i'|.$$

We therefore have

$$(5.3) \qquad \int_z^{\triangle} |\zeta_i'(y)/\zeta_i(y)|\,dy \leq \int_z^{\triangle} \left||\zeta_i|' / |\zeta_i|\right| dy + \int_z^{\triangle} |\phi_i'|\,dy.$$

Now let the numbers z_ν be all the points at which $|\zeta_i|'$ changes sign in the range of integration, except for two values of ν at which z_ν is equal to z and $\triangle$, and let Γ_ν be the larger of the two ratios

$$|\zeta_i(z_{\nu+1})| / |\zeta_i(z_\nu)| \quad \text{and} \quad |\zeta_i(z_\nu)| / |\zeta_i(z_{\nu+1})|.$$

We define a quantity $\delta\zeta_i$ which might be called the total variation of ζ_i, to be the product of all the Γ_ν:

$$\delta\zeta_i = \prod_\nu \Gamma_\nu .$$

Also, let the numbers z_λ be all the points at which ϕ_i' changes sign in the range of integration except for two at which z_λ equals z and $\triangle$. Then, by (5.3),

$$\int_{z}^{\triangle} |\zeta_i'(y)/\zeta_i(y)|\,dy \le \log(\delta\zeta_i) + \sum_{\lambda} |\phi(z_{\lambda+1}) - \phi(z_\lambda)|. \tag{5.4}$$

The expression (5.2) can now be written

$$\varepsilon \le |u_i(\triangle)| \left\{ (\delta\zeta_i)^{(1/\sqrt{2})} \exp\left[(1/\sqrt{2}) \sum_{\lambda} |\phi(z_{\lambda+1}) - \phi_i(z_\lambda)|\right] \right.$$
$$\left. \left[1 + (1/\sqrt{2})\log(\delta\zeta) + (1/\sqrt{2}) \sum_{\lambda} |\phi_i(z_{\lambda+1}) - \phi_i(z_\lambda)|\right] \right\}. \tag{5.5}$$

The case of zero conductivity is the case in which ϕ_i' is zero. Then we have simply

$$\varepsilon \le |u_i(\triangle)| \left\{ (\delta\zeta_i)^{(1/\sqrt{2})} - 1 - (1/\sqrt{2})\log(\delta\zeta) \right\}. \tag{5.6}$$

The inequality (5.6) can be replaced by a cruder but simpler estimate of the error. We have for $x \ge 0$

$$1/(x+1) \ge 1 - x;$$

after integrating both sides we have

$$\log(x+1) \ge x - x^2/2. \tag{5.7}$$

From (5.6) and (5.7) we obtain

$$\varepsilon \le |u_i(\triangle)| \left\{ (\delta\zeta)^{(1/\sqrt{2})} - 1 \right\}^2 / 2. \tag{5.8}$$

The exponent $1/\sqrt{2}$ in (5.8) can be left out in a slightly cruder estimate.

6. Error estimate for the reflection coefficient

We are more interested in computing the reflection coefficient, which is the ratio of D_i to U_i, than we are in the actual field quantities as determined by an initial value problem. This means that we should also investigate the error in the reflection coefficient due to the existence of an error in the field quantities. The reflection coefficient R_i is given by

$$R_i = D_i/U_i. \tag{6.1}$$

Let D_{io}/U_{io} be the approximate values of D_i and U_i based on the approximations (3.5) for d_i and u_i. Let ε_1 be the difference $u_i - u_{io}$ and ε_2 the difference $d_i - d_{io}$. Let D_{io}/U_{io} be our approximate reflection coefficient. We have for the error in computing R_i, assuming zero conductivity,

$$\begin{aligned}\varepsilon_i = |R_i - R_{io}| &= |D_i/U_i - D_{io}/U_{io}| = |(D_iU_{io} - D_{io}U_i)/U_iU_{io}| \\ &= |d_iu_{io} - u_id_{io}|/|u_iu_{io}| = |(d_{io}+\varepsilon_2)u_{io} - (u_{io}+\varepsilon_1)d_{io}|/|u_iu_{io}| \\ &= |u_{io}\varepsilon_2 - d_{io}\varepsilon_1|/|u_iu_{io}| = |\varepsilon_2/u_i - R_{io}\ \varepsilon_1/u_i| \\ &\leq |\varepsilon_2|/|u_i| + |R_{io}||\varepsilon_1|/|u_i| \leq |\varepsilon_2|/\left\{|u_{io}| - |\varepsilon_1|\right\} \\ &\qquad + |R_{io}||\varepsilon_1|\ /\left\{|u_{io}| - |\varepsilon_1|\right\}.\end{aligned} \tag{6.2}$$

Since every term in (6.2) contains or is bounded by a quantity containing $u_i(\triangle)$ as a factor we can set $u_i(\triangle) = 1$. Then

$$\varepsilon_r \leq |\varepsilon_2|\ /\left\{1 - |\varepsilon_1|\right\} + |R_{io}||\varepsilon_1|\ /\left\{1 - |\varepsilon_1|\right\}. \tag{6.3}$$

A difficulty seems apparent in (6.3) in case the approximate reflection coefficient R_{io} has a magnitude of the same order as the error ε_2, for then the error term may be large enough to obscure any significant approximation of the reflection coefficient. It is therefore necessary to determine whether $|\varepsilon_2|$ is sufficiently small to justify assuming that R_i is zero in this case.

We observe that in the electromagnetic problem the matrix $A_{ij}(z)$ of our system of integral equations has only zero elements in its main diagonal. The product of two such matrices is a diagonal matrix. It follows that if we perform another iteration of (3.4) to obtain a correction to (3.5), the approximation to the component $d_i(z)$ remains unchanged while $u_i(z)$ gets a correction term added to it. Our conclusion is that the estimate of the error in our approximation to $d_i(z)$ is of higher order than we have actually assumed, corresponding in fact to one more iteration of the integral equation, although our error estimate for the component $u_i(z)$ is more or less correct. Thus, $|\varepsilon_2|$ is bounded by a quantity of higher order than the bound on $|\varepsilon_1|$. Thus, according to (6.3), the error ε_r will either be of higher order than R_i or else we may set R_i equal to zero with an error of higher order than $|\varepsilon_1|$. Since $|\varepsilon_1|$ is of higher order than the amplitude of the incident wave, we conclude that our approximation to R_i is significant, and a significant bound on the error commited in this approximation is given by

$$\varepsilon_r \leq \varepsilon^2/2(1-\varepsilon) + |R_{io}|\varepsilon/(1-\varepsilon) \tag{6.4}$$

where ε is given by (5.5) or (5.8). Implicit in this result is the assumption that ε is less than one as, in fact, it must be in order that the approximation have any real meaning.

When the conductivity is different from zero the error estimate (6.4) should be multiplied by the exponential factor

$$\left| \exp\left\{ -j \int_z^{\triangle} k_i(s)ds \right\} \right| , \tag{6.5}$$

which is obviously one when $k_i(s)$ is real in the interval from z to $\triangle$.

7. Summary and conclusions

We have treated approximately the reflection problem arising from a plane electromagnetic wave incident on a plane stratified inhomogeneous medium, and we have obtained a bound on the error committed in the approximation. The perturbation method employed in this treatment can be characterized roughly by saying that it holds for a medium of small variation. A precise meaning is given to the term " small variation" if we define it to be equivalent to the statement that the quantity $\delta\zeta_i$ (defined in Section 5 to be the total variation of ζ_i) is small.

Two rather unusual features of the approximation treated here should be noted. First, it is valid for all frequencies ω to within the error ε_r (defined in Section 6) which is independent of frequency. Second, it is valid even when derivatives of the electrical parameters ε, μ and σ of the medium have jump discontinuities.

For easy reference a summary of our treatment, including the relevant formulas, follows:

(1) *We assume a cartesian coördinate system whose z-axis is normal to the planes on which the electrical parameters are constant. The x,y-plane bounds the inhomogeneous medium below, while a plane* $z = \triangle$ *bounds it above. The y,z-plane is the plane of incidence, and the angle of incidence is designated by* α *(see Figure 1).*

(2) *Letting the subscript p stand for polarization parallel to the plane of incidence and n for polarization normal to the plane of incidence, we write for the field vectors E and H*

$$E = E_p + E_n, \qquad H = H_p + H_n,$$

where the vectors have the form

$$E_p = (0, V_p, \mathcal{E}) \exp(-jk_o y \sin\alpha),$$

$$H_p = (I_p, 0, 0) \exp(-jk_o y \sin\alpha),$$

$$E_n = (V_n, 0, 0) \exp(-jk_o y \sin\alpha),$$

$$H_n = (0, I_n, \mathcal{L}) \exp(-jk_o y \sin\alpha),$$

with $k_o = \omega\sqrt{\varepsilon_o\mu_o}$. *Inductances* L_p, L_n *and capacitances* C_p, C_n *are defined by*

$$L_p(z) = (\mu_o\varepsilon_o\sin^2\alpha)/\ \hat{\varepsilon}(z)-\mu(z),$$

$$L_n(z) = \mu(z),$$

$$C_p(z) = -\hat{\varepsilon}(z),$$

$$C_n(z) = \hat{\varepsilon}(z) - (\mu_o\varepsilon_o\sin^2\alpha)/\mu(z).$$

(3) *Equations for the field vector components are*

$$dV_i/dz = -j\omega L_i(z)I_i,$$

$$dI_o/dz = -j\omega C_i(z)V_i, \qquad i = p,n,$$

along with

$$\mathcal{E}(z) = \left\{(k_o\sin\ \alpha)/\ \omega\hat{\varepsilon}\right\}\ I_p(z),$$

$$\mathcal{H}(z) = \left\{(-k_o\sin\ \alpha)/\ \omega\mu\right\}\ V_n(z).$$

(4) *The characteristic impedances* $\zeta_i(z)$ *are defined by*

$$\zeta_i(z) = \left\{L_i(z)/C_i(z)\right\}^{1/2}, \qquad i = p,n,$$

and the wave numbers $k_i(z)$ *by*

$$k_i(z) = \omega\left\{L_i(z)C_i(z)\right\}^{1/2}, \qquad i = p,n.$$

(5) *The upgoing components* $U_i(z)$ *and the downgoing components* $D_i(z)$ *of the traveling wave representation are determined by the relations*

$$V_i(z) = U_i(z) + D_i(z),$$

$$I_i(z) = \left\{U_i(z) - D_i(z)\right\}\ /\ \zeta_i(z), \qquad i = p,n.$$

(6) *The perturbation factors* $u_i(z)$ *and* $d_i(z)$ *for these quantities are given by the relations*

$$U_i(z) = u_i(z)\, \zeta^{1/2}(z)\, \exp\left\{ j \int_z^{\Delta} k_i(s)ds \right\}$$

$$D_i(z) = d_i(z)\, \zeta_i^{1/2}(z)\, \exp\left\{ -j \int_z^{\Delta} k_i(s)ds \right\}, \qquad i = p,n.$$

(7) *The approximate solution is given by*

$$u_i(z) \sim u_i(\Delta)$$

$$d_i(z) \sim u_i(\Delta) \int_z^{\Delta} \left\{ \zeta_i'(y)/2\zeta_i(y) \exp\left[2j \int_y^{\Delta} k_i(s)ds \right] \right\} dy, \qquad i = p,n.$$

(8) *Let the numbers* z_ν *be the points at which the derivative* $|\zeta_i(z)|'$ *changes sign, except for two values of* z_ν *which are* z *and* Δ *and let* Γ_ν *be the greater of the two ratios*

$$|\zeta_i(z_{\nu+1})| \,/\, |\zeta_i(z_\nu)| \quad \text{and} \quad |\zeta_i(z_\nu)| \,/\, |\zeta_i(z_{\nu+1})|.$$

Then the total variation $\delta\zeta_i$ *of* ζ_i *is defined to be*

$$\delta\zeta_i = \prod_\nu \Gamma_\nu .$$

(9) *Let* $\phi_i(z)$ *be the phase of* $\zeta_i(z)$ *and the numbers* z_λ *the points where the derivative* $\phi_i'(z)$ *changes sign, except for two values of* z_λ *which are* z *and* Δ. *A bound on the error* ε_i *in our approximation to the vector* (u_i, d_i) *is given by*

$$\varepsilon_i \leq (\delta\zeta_i)^{(1/\sqrt{2})} \exp\left\{ (1/\sqrt{2}) \sum_\lambda |\phi_i(z_{\lambda+1}) - \phi_i(z_\nu)| \right\}$$

$$- \left\{ 1 + (1/\sqrt{2})\left(\log \delta\zeta_i + \sum_\lambda |\phi_i(z_{\lambda+1}) - \phi_i(z_\lambda)|\right) \right\}, \qquad i = p,n.$$

(10) *If we assume that the conductivity is zero then*

$$\varepsilon_i \leq (\delta\zeta_i)^{(1/\sqrt{2})} - \left\{1 + (1/\sqrt{2})\log(\delta\zeta_i)\right\}, \qquad i = p,n,$$

a cruder estimate is

$$\varepsilon_i \leq \left\{(\delta\zeta_i)^{(1/\sqrt{2})} - 1\right\}^2/2 \leq \left\{\delta\zeta_i - 1\right\}^2/2, \qquad i = p,n.$$

(11) *A bound on the error estimate* ε_r *for the reflection coefficient is*

$$\varepsilon_{ri} \leq \left\{\varepsilon_i^2/2(1-\varepsilon_i) + |R_{io}|\varepsilon_i/(1-\varepsilon_i)\right\} \exp\left\{-j\int_z^{\Delta} k_i(s)ds\right\}, \qquad i = p,n,$$

where R_{io} *is our approximate value for the reflection coefficient determined by the ratio*

$$R_{io} = D_{io}/U_{io}, \qquad i = p,n$$

of our approximations D_{io} *and* U_{io} *to* D_i *and* U_i.

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